Computers and Mathematics with Applications 60 (2010) 390–398

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A multiscale finite element method for optimal control problems governed by the elliptic homogenization equations

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ARTICLE INFO

Article history:
Received 18 April 2009
Received in revised form 8 April 2010
Accepted 13 April 2010

Keywords:
Finite element methods
A multiscale finite element method
Heterogeneous theory
Optimal control problems
A priori estimate

ABSTRACT

In this article, we develop and analyze a priori estimates for optimal control problems with multiscale governed by the elliptic homogenization equations. The multiscale finite element is applied to capture the effect of microscale through modification of finite element basis functions without resolving all the small scale features. The optimal estimate is derived for elliptic homogenization problems without resonance effect $O(\epsilon/h)$ by using an over-sampling technique and the boundary layer assumption. Furthermore, the a priori estimate is obtained for the optimal control problems governed by the elliptic homogenization equations.

1. Introduction

Flow control problems with multiple scale are fundamental and of practical importance in science and engineering. Multiscale problems in science and engineering are often described by partial differential equations with highly oscillatory coefficients. Typical examples include porous media and composite materials. Resolving different scales with good accuracy in reasonable computational time is a very challenging task. Although tremendous amount of computer memory and *CPU* time is saved by using parallel computing to some degree, the size of computation is not reduced in the traditional approaches which directly solve the equations on fine meshes. In flow control problems, boundary and shape control are widely used, though distributed control is also available through a magnetic field, a heat source using radiation, or laser technology. Most importantly, more attention has been paid to the control problem with different application background. Therefore, various methods have been developed for the multiscale problem and the control optimal problem.

This article concentrates on the multiscale finite element method for the elliptic homogenization problem. In the past years, several related numerical techniques have been studied. Examples includes wavelet homogenization techniques [1], multigrid numerical homogenization techniques [2–4], the multiscale finite element method [5–7], finite element method based on the Residual-Free Bubble method, and the heterogeneous multiscale method [8]. The purpose of this article is to present a multiscale finite element method [9,5–7], which combines the multiscale finite element method and the heterogeneous multiscale method. This method can be applied to a large variety of differential problems and numerical methods easily. This is accomplished by constructing the multiscale finite element base functions that are adaptive to the local property of the differential operator. Also, the method is perfectly parallel and is naturally adapted to massively parallel computers. It is mentioned that we propose an over-sampling technique and the assumption of the numerical results with respect to boundary layer to remove the resonance effect $O(\epsilon/h)$ in this article.

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doi:10.1016/j.camwa.2010.04.017

Furthermore, we analyze error estimates of the optimal control problems governed by the elliptic homogenization equations. Presently, the finite element method is the most widely used numerical method in optimal control problems. Systematic introduction of the finite element method for PDEs and optimal control problems can be found in [10–13]. These techniques have been developed for finite element approximation of various optimal control problems. Especially, a priori error estimate of finite element approximation have been established for the optimal control problems governed by linear elliptic or parabolic state equations [14–18]. Then, the estimates indicate the relationship between the mesh scales of the state problem and the control problems in theory. Admittedly, there are a lot of wonderful research, which can guide us to design two families of mesh scalings both for the control problems and the state problems. In this way, much unnecessary consumption is saved in practice.

The remainder of this paper is organized as follows. In the next section, the results for elliptic homogenization equations are recalled, together with some basic notations. The multiscale finite element method is used to obtain optimal error estimate for the elliptic homogenization equation in Section 3. Finally, the a priori estimate is derived for the control problems governed by the elliptic homogenization equations.

2. The standard finite element method

This section considers the second order elliptic model problem with highly oscillatory coefficients

$$-\mathrm{div}(a_\epsilon \nabla u_\epsilon) = f, \quad \text{in } \Omega, \tag{2.1}$$

$$u_\epsilon = 0, \quad \text{on } \partial\Omega, \tag{2.2}$$

where Ω be a bounded domain in R^d, $d = 2, 3$, with Lipschitz boundary Γ; ϵ is a small parameter; $a_\epsilon = a_\epsilon(y)$ is symmetry and satisfies the following property

$$\alpha|\xi|^2 \le \xi_i a_{ij}^\epsilon \xi_j \le \beta|\xi|^2, \quad \forall \xi \in R^2, \tag{2.3}$$

with the relations $0 < \alpha < \beta < \infty$ and $y = x/\epsilon$. Moreover, we assume that $a_\epsilon(y)$ is quasi-continuous in y uniformly with respect to x and ϵ and periodic in y with period $I = [0, 1]^d$.

For convenience, let

$$X = H_0^1(\Omega), \qquad H = L^2(\Omega), \qquad U = L^2(\Omega_U),$$

where $\Omega_U \subset \Omega$. Variational problem of (2.1)–(2.2) is to seek $u_\epsilon \in X$ s.t.

$$a(u_\epsilon, v) = (f, v), \tag{2.4}$$

where

$$a(u_\epsilon, v) = \int_\Omega a_{ij}^\epsilon \frac{\partial u_\epsilon}{\partial x_i}\frac{\partial v}{\partial x_j}\mathrm{d}x, \qquad (f, v) = \int_\Omega f v \mathrm{d}x.$$

Obviously, the existence and uniqueness of (2.4) can be easily obtained by its elliptic and continuous properties.

Let τ_h be a family of quasi-uniform triangulations of a bounded polygonal domain Ω. In particular, h_K denotes the diameter of the triangle $K \in \tau_h$, and $h = \max_K\{h_K\}$. Assume that X_h be finite element space approximated by the piecewise linear finite element. The standard finite element analysis presented the following approximation property for any $v \in X$

$$\|v - v_h\|_{0,K} + h\|v - v_h\|_{1,K} \le Ch^2\|v\|_{2,K}, \quad \forall v_h \in X_h. \tag{2.5}$$

Moreover, the following result is concerned with the regularity of the solution to (2.1)–(2.2)

$$\|u^\epsilon\|_{1,\Omega} \le C\|f\|_{0,\Omega}, \qquad \|u^\epsilon\|_{2,\Omega} \le C\epsilon^{-1}\|f\|_{0,\Omega}. \tag{2.6}$$

Then, the error analysis is given for the elliptic homogenization problems as follows

$$\|u^\epsilon - u_h\|_{0,\Omega} + h\|u^\epsilon - u_h\|_{1,\Omega} \le C\epsilon^{-1}h^2\|f\|_{0,\Omega}. \tag{2.7}$$

Note that $C > 0$ is a constant depending only on Ω. Below the constant $C_i > 0, i = 0, 1, 2\cdots$ will depend at most on the data (Ω, f). Standard definitions are used for the Sobolev spaces $W^{m,r}(\Omega)$, with the norm $\|\cdot\|_{W^{m,r}}$ and the seminorm $|\cdot|_{W^{m,r}}$, $m, r \ge 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_{H^m}$ for $\|\cdot\|_{W^{m,2}(\Omega)}$. Especially, due to the norm equivalence between $\|u\|_1$ and $\|\nabla u\|_0$ on $H_0^1(\Omega)$, we are using the same notation for them.

3. The multiscale finite element method

The main technique of the mixed multiscale finite element method is to construct the modify finite element basis functions, which capture the microscale information on macroscale without solving all the fine scale structures [8,5–7]. We only utilize the modified finite element basis with multiscale information to carry out computation on the advisable mesh. Thus, we only dwell on the microscale computation for the homogenization problems.

The modified finite element basis functions is to seek $R(v) \in H^1(K),\ K \in K_h$ such that

$$(a_\epsilon R(v), w)_K = 0, \quad \forall w \in K, \tag{3.1}$$

$$R(v) = v, \quad \text{on } \partial K. \tag{3.2}$$

Note that the operator R is defined on the finite element set K.

Then, the multiscale finite element method for (2.4) is to find $u_\epsilon^h \equiv R(u_h) \in X_h$ such that

$$(a_\epsilon \nabla u_\epsilon^h, \nabla v_\epsilon^h) = (f, v_\epsilon^h), \quad \forall v_\epsilon^h \in X_h. \tag{3.3}$$

It is easy to obtain the existence and uniqueness of a solution to (3.3) by the following continuous and coercive properties and the Lax–Milgram theorem

$$|(a_\epsilon \nabla u_\epsilon^h, \nabla v_\epsilon^h)| \le C\|u_h\|_{1,\Omega}\|v_h\|_{1,\Omega} \quad \forall u_h, v_h \in X_h, \tag{3.4}$$

$$|(a_\epsilon \nabla u_\epsilon^h, \nabla u_\epsilon^h)| \ge C\|u_h\|_{1,\Omega}^2 \quad \forall u_h \in X_h, \tag{3.5}$$

Thus, two functions satisfy the following relation:

$$C_1\|v_h\| \le \|v_\epsilon^h\|_{1,\Omega} \le C_2\|v_h\|_{1,\Omega}, \quad \forall v_h \in X_h. \tag{3.6}$$

Unfortunately, the standard finite element analysis for (3.3) is also dependent of the order $O(h/\epsilon)$. From the point of view of computation, it is required to satisfy $h \ll \epsilon$ for mesh scaling relation between h and ϵ in order to guarantee the optimal convergence of (2.7). Therefore, lots of computation had to perform in order to obtain the certain accuracy in practice for the elliptic problem with highly oscillatory coefficients.

The homogenization theory has been used in the analysis of the multiscale finite element method in [19–21]. The theory provides the detailed structures of the physical solution and the multiscale basis. Moreover, the homogenization theory provides an effective way to avoid the worse convergence order $O(h/\epsilon)$.

Let $U_0 \in H^2(\Omega)$ be the solution to the homogenized problem

$$(A_{ij}\nabla U_0, \nabla v) = (f, v) \quad \forall v \in X, \tag{3.7}$$

where the constant homogenized coefficient a_{ij}^* are given by

$$A_{ik} = \frac{1}{|I|}\int_I a_{ij}(y)\left(\delta_{jk} + \sum_{k=1}^{d}\frac{\partial \chi_k}{\partial y_j}(y)\right)\mathrm{d}y$$

and $\chi_{(y)}$ is the periodic solution to the cell problem

$$-\mathrm{div}(a_\epsilon \nabla_y \chi^j) = \sum_{i=1}^{d}\frac{\partial}{\partial y_i}a_{ij}(y) \quad y \in I, \tag{3.8}$$

$$\int_I \chi^j(y)\mathrm{d}y = 0. \tag{3.9}$$

A brief review of the multiscale expansion of the solution can be expanded as follows

$$u_1^\epsilon = U_0 + \epsilon\sum_{k=1}^{d}\chi^k(y)\frac{\partial U_0}{\partial x_k}. \tag{3.10}$$

Obviously, the first order correction $\theta_\epsilon = u^\epsilon - u_1^\epsilon$ can be given by

$$-\mathrm{div}(\nabla(a_\epsilon \nabla\theta_\epsilon)) = 0, \quad \text{in } K, \tag{3.11}$$

$$\theta_\epsilon = -\epsilon\sum_{k=1}^{d}\chi^k(y)\frac{\partial U_0}{\partial x_k}, \quad \partial K. \tag{3.12}$$

It is well known that the classical results can be presented in [7,22].

Lemma 3.1. *Under the assumption of $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and $f \in L^2(\Omega)$, let u_ϵ and U_0 be the solution of (2.1)–(2.2) and (3.7), respectively. Then, it holds*

$$\|u_\epsilon - U_0\|_{1,\Omega} \le C\epsilon \|U_0\|_{2,\Omega}, \tag{3.13}$$

$$\|u_\epsilon - u_1^\epsilon\|_{1,\Omega} \le C\sqrt{\epsilon}\|U_0\|_{1,\infty,\Omega}. \tag{3.14}$$

Furthermore, we have

$$\|u^\epsilon - u_\epsilon^h\|_{1,\Omega} \le C\left((h+\epsilon)\|f\|_{0,\Omega} + \sqrt{\frac{\epsilon}{h}}\|U_0\|_{1,\infty,\Omega}\right), \tag{3.15}$$

$$\|u^\epsilon - u_\epsilon^h\|_{0,\Omega} \le C\left((h^2+\epsilon)\|f\|_{0,\Omega} + \sqrt{\frac{\epsilon}{h}}\|U_0\|_{1,\infty,\Omega}\right). \tag{3.16}$$

Obviously, the results of (3.15)–(3.16) deteriorates as $h = O(\epsilon)$. In other words, the above results suffer from the resonance error by using the multiscale finite element method. The main cause is that the resonance error has a boundary layer structure [6]. The usual way is to apply the over-sampling method in order to reduce the resonance error. Detailedly, we construct the modified base functions $R(v)$ on a sampling domain $S \supset K$ with $\mathrm{diam}(S) = H > h$ by solving the following problem

$$(a_\epsilon \nabla R(v), \nabla w)_S = 0, \quad \text{in } S, \tag{3.17}$$

$$R(v)|_{\partial S} = v, \quad \text{on } \partial S. \tag{3.18}$$

Moreover, ∂S is away from ∂K at some distance dS. Then, the finite element basis on K is constructed by the following procedure. Assume that $\{\phi_i^K\}_1^{d+1}$ and $\{\psi_j^S\}_1^{d+1}$ are the basis of $X_h(K)$ and $X_h(S)$, respectively, then the relation between two kinds of basis is presented for $\forall v \in X(K)$ as follows

$$v|_K = \sum_{i=1}^{d+1} c_i^K \phi_i^K, \qquad \phi_i^K = \sum_{i=1}^{d+1} c_{ij}^K \psi_j^S|_K. \tag{3.19}$$

Thus, we can obtain all the basis function on each finite element K. For convenience, the global operator R is defined by

$$v_\epsilon^h = R(v_h)|_K = R_S(v_h)|_K, \quad \forall v_h \in X_h, K \in \tau_h.$$

Obviously, the continuity across the internal boundaries of the finite element $K \subset \tau_h$ cannot be guaranteed by using the over-sampling technique. Therefore, we define the norm

$$\|v\|_{h,\Omega} = \left(\sum_{K\in K_h} \|\nabla v\|_{0,K}^2\right)^{1/2}, \quad v \in X_h.$$

Clearly, it is easy to show that $\|v\|_{h,\Omega}$ is a norm. Then, the multiscale finite element method for (2.4) is given by

$$a_h(u_\epsilon^h, v_\epsilon^h) = (f, v_\epsilon^h), \quad \forall v_\epsilon^h \in X_h, \tag{3.20}$$

where $a_h(v_\epsilon^h, w_\epsilon^h) \equiv \sum_{K\in K_h}(a_\epsilon \nabla v_\epsilon^h, \nabla w_\epsilon^h)$.

Thus, the Céa Lemma is replaced by the following second Strang's Lemma.

Lemma 3.2. *Let u_ϵ^h and u_ϵ are the solutions to (2.1)–(2.2) and (3.20). Then, we have*

$$\|u_\epsilon - u_\epsilon^h\|_{h,\Omega} \le C\left\{\inf_{v\in X_h}\|u_\epsilon - v\|_{h,\Omega} + \sup_{w_\epsilon^h \in X_h}\frac{\sum_{K\in\tau_h}[a_h(u_\epsilon, w_\epsilon^h) - (f, w_\epsilon^h)]}{\|w_\epsilon^h\|_{1,\Omega}}\right\}. \tag{3.21}$$

The first term in the right hand side is referred to as the approximation error, and the second term is called the consistency error.

According to homogenization theory and finite element theory, we define the interpolation of (3.10) by

$$I_h u_1^\epsilon = I_h U_0 + \epsilon \sum_{k=1}^d \chi^k(y)\frac{\partial I_h U_0}{\partial x_k}. \tag{3.22}$$

Similarly, the first order correction $I_h\theta_\epsilon(v) = -\epsilon \sum_{k=1}^d \eta^k(y)\frac{\partial I_h U_0}{\partial x_k}$ satisfies the following problem for $v \in X_h$

$$-\nabla\cdot(a_\epsilon \nabla\theta_\epsilon) = 0, \quad \text{in } S, \tag{3.23}$$

$$\eta^k = -\chi^k, \quad \text{on } \partial S. \tag{3.24}$$

From [6], dS is determined by the thickness of the boundary layer of η. Numerically, it has been observed that the boundary layer is about $O(\epsilon)$ thick. The mathematical result shows that $dS = h(> \epsilon)$ is usually sufficient for eliminating the boundary layer effect. In detail, we assume the following assumption.

Lemma 3.3 *([7]). Let $K \subset S$ for any $K \in \tau_h$. There holds*

$$\|\nabla\eta\|_{0,\infty,K} \le C, \tag{3.25}$$

where C is independent of h_K and ϵ.

In the following, we will provide the results for the elliptic homogenization problems by using the multiscale finite element method.

Lemma 3.4. *Under the assumption of $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and (2.6), we have*

$$\inf_{v\in X_h} \|u_\epsilon - v\|_{h,\Omega} \le C(\epsilon + h). \tag{3.26}$$

Proof. By the definition of u_1^ϵ, we have

$$\nabla u_1^\epsilon = \nabla U_0 + \sum_{k=1}^{d}\left(\nabla_y \chi^i(y)\frac{\partial U_0}{\partial x_i} + \epsilon\chi^i(y)\nabla\frac{\partial U_0}{\partial x_i}\right).$$

Thus, we deduce from the Cauchy–Schwarz inequality that

$$\|u_1^\epsilon - I_h u_1^\epsilon\|_{h,\Omega} \le C(\epsilon + h). \tag{3.27}$$

Applying the maximum principle and Lemmas 3.1–3.3, yields that

$$\begin{aligned}\|\nabla\theta_\epsilon(v)\|_{0,S} &\le C\epsilon(\|\nabla\eta^k\|_{0,\infty,S}\|I_hU_0\|_{1,S} + \|\eta^k\|_{0,\infty,S}\|I_hU_0\|_{2,S})\\ &\le C\epsilon,\end{aligned} \tag{3.28}$$

since a_ϵ and χ^k are smooth functions. Using the triangle inequality, we derive from (3.14) and (3.27)–(3.28) that

$$\begin{aligned}\inf_{v\in X_h}\|u_\epsilon - v\|_{h,\Omega} &\le \|u_\epsilon - I_hu_\epsilon\|_{h,\Omega}\\ &\le \|u_\epsilon - u_1^\epsilon\|_{h,\Omega} + \|u_1^\epsilon - I_hu_1^\epsilon\|_{h,\Omega} + \|\theta_\epsilon(v)\|_{h,\Omega}\\ &\le C(\sqrt{\epsilon} + h).\end{aligned} \tag{3.29}$$

As to the consistency error, we have the following results. □

Lemma 3.5. *Let $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and (2.6), we have*

$$\sup_{w_h\in X_h}\frac{\sum_{K\in\tau_h}[(a_\epsilon\nabla u_\epsilon, \nabla w_h) - (f, w_h)]}{\|w_h\|_{1,\Omega}} \le C(\sqrt{\epsilon} + h). \tag{3.30}$$

Proof. In order to obtain the consistency error, we apply some useful results in [9,5–7]. Thanks to the norm definition and the inverse inequality, and (3.23)–(3.24), we have

$$\begin{aligned}\|\nabla\theta_\epsilon\|_{0,K} &\le ch_K^{-1}\|\theta_\epsilon\|_{0,S}\\ &\le ch_K^{-1}\|\eta\|_{0,S}\|I_hU_0\|_{0,\infty,S}\\ &\le c\epsilon h_K^{d/2-1}\|U_0\|_{1,\infty,K}.\end{aligned} \tag{3.31}$$

Also, by the definition of $\chi(y)$ in (3.8)–(3.9), we arrive at

$$\sum_{j=1}^{d+1} a_{ij}\frac{\partial u_1^\epsilon}{\partial x_j} = \sum_{j=1}^{d+1}\left(a_{ij} + \sum_{\ell=1}^{d+1}\left(a_{i\ell}\frac{\partial\chi^j}{\partial x_\ell}\right)\right)\frac{\partial U_0}{\partial x_k} + \epsilon\sum_{j=1}^{d+1}(a_{ij}\chi^k)\frac{\partial^2 U_0}{\partial x_k\partial x_j} \tag{3.32}$$

and

$$\int_I\left(a_{ij} + \sum_{\ell=1}^{d+1}\left(a_{i\ell}\frac{\partial\chi^j}{\partial x_\ell}\right)\right)\nabla_y\chi^j \mathrm{d}x = 0. \tag{3.33}$$

Furthermore, applying the technique of homogenization theory, and noting that assumption in [6] for any $w \in H^1(K) \cap L^\infty(K)$

$$\int_I v \mathrm{d}x = 0 \quad \forall v \in L^\infty(K),$$

yields that

$$\left| \int_K v(x/\epsilon) w(x) \mathrm{d}x \right| \le C\epsilon (h_K^{d/2} |w|_{1,K} + h_K^{d-1} \|w\|_{0,\infty,K}). \tag{3.34}$$

As for the truncation error, noting that

$$v_\epsilon^h = v + \epsilon \sum_{k=1}^d \chi^k \frac{\partial v}{\partial x_k} + \theta_\epsilon(v),$$

substituting the definition of v_ϵ^h and homogenization theory that

$$\begin{aligned}(a_\epsilon \nabla u_\epsilon, \nabla v_\epsilon^h)_K - (f, v_\epsilon^h)_K = {} & \left[(a_\epsilon \nabla u_\epsilon, \nabla v) + \left(a_\epsilon \nabla u_\epsilon, \nabla \left(\epsilon \sum_{k=1}^d \chi^k \frac{\partial v}{\partial x_k} \right) \right) + (a_\epsilon \nabla u_\epsilon, \nabla \theta_\epsilon(v)) \right]_K \\ & - \left[(f, v) + \left(f, \nabla \left(\epsilon \sum_{k=1}^d \chi^k \frac{\partial v}{\partial x_k} \right) \right) + (f, \theta_\epsilon(v)) \right]_K . \end{aligned} \tag{3.35}$$

Obviously, it follows from (2.4), (2.6) and (3.32) [6] that

$$\sum_{K \subset \tau_h} [(a_\epsilon \nabla u_\epsilon, \nabla v) - (f, v)] = 0, \tag{3.36}$$

$$\begin{aligned} |(a_\epsilon \nabla u_\epsilon, \nabla \theta_\epsilon(v))_K| & \le c\epsilon h_K^{d/2-1} \|u_\epsilon\|_{1,K} \|v\|_{1,\infty,K} \\ & \le c\epsilon h_K^{d/2-1} \|f\|_{0,\Omega} \|v\|_{1,\infty,K}, \end{aligned} \tag{3.37}$$

$$\left| \left(f, \epsilon \sum_{k=1}^d \chi^k \frac{\partial v}{\partial x_k} \right)_K \right| \le c\epsilon \|f\|_{0,\Omega} \|v\|_{1,K}, \tag{3.38}$$

$$\begin{aligned} |(f, \theta_\epsilon(v))_K| & \le c \|f\|_{0,\Omega} \|\theta_\epsilon(v)\|_{0,S} \\ & \le c\epsilon \|f\|_{0,\Omega} \| \sum_{k=1}^d \chi^k \frac{\partial v}{\partial x_k} \|_{0,\partial S} \\ & \le c\epsilon h_K^{(d-1)/2} \|f\|_{0,\Omega} \|\nabla \chi^k\|_{0,\infty,K} \|v\|_{1,\infty,K}. \end{aligned} \tag{3.39}$$

By homogenization theory, we arrive at

$$\begin{aligned} & \left(a_\epsilon \nabla u_\epsilon, \nabla \left(\epsilon \sum_{k=1}^d \chi^k \frac{\partial v}{\partial x_k} \right) \right)_K = \left(a_\epsilon \nabla u_\epsilon, \sum_{k=1}^d \nabla \chi^k \frac{\partial v}{\partial x_k} \right)_K + 0 \\ & = \left(a_\epsilon \nabla [u_\epsilon - u_1^\epsilon], \sum_{k=1}^d \nabla \chi^k \frac{\partial v}{\partial x_k} \right)_K + \left(a_\epsilon \nabla u_1^\epsilon, \sum_{k=1}^d \nabla \chi^k \frac{\partial v}{\partial x_k} \right)_K \\ & = \left(a_\epsilon \nabla [u_\epsilon - u_1^\epsilon], \sum_{k=1}^d \nabla \chi^k \frac{\partial v}{\partial x_k} \right)_K + \left(\left(a_{ij} + \sum_{\ell=1}^d \left(a_{i\ell} \frac{\partial \chi^j}{\partial x_\ell} \right) \right) \frac{\partial U_0}{\partial x_k}, \sum_{k=1}^d \nabla \chi^k \frac{\partial v}{\partial x_k} \right)_K \\ & \quad + \left(\epsilon \sum_{j=1}^d (a_{ij} \chi^k) \frac{\partial^2 U_0}{\partial x_k \partial x_j}, \sum_{k=1}^d \nabla \chi^k \frac{\partial v}{\partial x_k} \right)_K \\ & = I_1 + I_2 + I_3. \end{aligned} \tag{3.40}$$

Thus, it follows from (3.25) that

$$|I_1| \le C \|u_\epsilon - u_1^e\|_{1,K} \|v\|_{1,K} \|\nabla \chi\|_{0,\infty,K}, \tag{3.41}$$

$$|I_2| \le C\epsilon \{ h_K^{d/2} |U_0|_{1,K} + h_K^{d-1} \|U_0\|_{0,\infty,K} \} \|v\|_{1,K}, \tag{3.42}$$

$$|I_3| \le C\epsilon \|U_0\|_{2,K} \|v\|_{1,K} \|\nabla \chi\|_{0,\infty,K}. \tag{3.43}$$

Combining all these inequalities with (3.36) and summing over all set K yields (3.30). □

Theorem 3.6. *Let $U_0 \in W^{1,\infty}(\Omega) \bigcap H^2(\Omega)$ and $f \in L^2(\Omega)$, we have*

$$\|u_\epsilon - u_\epsilon^h\|_{h,\Omega} \le C(\sqrt{\epsilon} + h). \tag{3.44}$$

Proof. Obviously, (3.44) follows from Lemmas 3.3–3.5. □

4. A priori estimate for optimal control problems

This section considers a priori estimates of the optimal control problems governed by the elliptic homogenization problems. As note in the previous sections, we have obtained the multiscale finite element method without resonance error. Here, the control problem is designed and a priori estimate will be shown for the optimal control problems governed by the elliptic homogenization equations.

Consider the optimal control problems:

$$\begin{cases} S(u) = \min_{v\in K\subset U}\{g(y) + h(u)\} \\ -\mathrm{div}(a_\epsilon \nabla u_\epsilon) = f + Bu, \quad \text{in } \Omega, \\ u_\epsilon = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The well-posedness of the model problem (4.1) then follows from Lions' theory for partial differential problems [11]. In particular, there exists a unique solution $(u_\epsilon, p_\epsilon, u)$ for the above optimal control problem. Moreover, $(u_\epsilon, p_\epsilon, u)$ is the solution of the control problem (4.1) if and only if there is a co-state $p_\epsilon \in X$ such that $(u_\epsilon, p_\epsilon, u)$ satisfies the following optimality conditions (4.2). Then the optimal control problem can be equivalent to the following variational problem [11]: Find $(u_\epsilon, p_\epsilon, u) \in (X \times X \times U)$ such that

$$\begin{cases} a(u_\epsilon, v) = (f + Bu, v), \quad \forall v \in X, \\ a(q, p_\epsilon) = (g'(u_\epsilon), q), \quad \forall q \in X, \\ (h'(u) + B^* p_\epsilon, w - u) \ge 0, \quad \forall w \in Z. \end{cases} \tag{4.2}$$

Here, the state space and control space will be X and U. The operator B is linear bounded and self-conjugated: $B: U \to H$. The set Z is a closed convex set in U. Also, $g'(y)$ and $h'(u)$ are Lipschitz continuous and uniformly convex in $H = H'$ and $U = U'$, respectively. Moreover, $h'(u) + B'p_\epsilon \in H^1(U)$.

Then, the discrete weak formula of the control problem (4.2) is to seek $(u_\epsilon^h, p_\epsilon^h, u_h) \in (X_h \times X_h \times U_h)$ read as

$$\begin{cases} a_h(u_\epsilon^h, v_\epsilon^h) = (f + Bu_h, v_\epsilon^h), \quad \forall v_h \in X_h, \\ a_h(q_h, p_\epsilon^h) = (g'(u_\epsilon^h), q_h), \quad \forall q_h \in X_h, \\ (h'(u_h) + B^* p_h, w_h - u_h) \ge 0, \quad \forall w_h \in U_h. \end{cases} \tag{4.3}$$

As we known, the optimal control problems consume much time in computing. Thus, it is efficient to solve it for optimal mesh distribution by using adaptive finite element method. Concretely, two families of partitions are constructed to deal with the optimal control problem. If the solution is smooth enough, the mesh for the state and the co-state equation is the uniform triangular mesh τ_h and the appropriate finite elements are adopted for the state and co-state equations. Also, another partition τ_{h_U} is designed for the control problems by piecewise constant because the regularity of the object function u is at most $H^1(\Omega_U)$.

Now, we has established the preliminary results of the optimal control problem for the elliptic homogenization problems. Then, a priori estimate is estimated by combining the multiscale finite element method and optimal control analysis for the elliptic homogenization problems.

Theorem 4.1. *Let X_h and U_h are piecewise linear finite element space and piecewise constant finite element space, respectively. Assume that $(u_\epsilon, p_\epsilon, u)$ and $(u_\epsilon^h, p_\epsilon^h, u_h)$ are solutions to problem (4.2) and (4.3), respectively. If $S(u)$ is uniformly convex at an neighborhood of the solution, it holds, for any $u, v \in Z$*

$$(S'(u) - S'(v), u - v)_{\Omega_U} \ge c\|u - v\|^2_{0,\Omega_U}, \tag{4.4}$$

then we have

$$\|u - u_h\|_{0,\Omega_U} + \|u_\epsilon - u_\epsilon^h\|_{h,\Omega} + \|p_\epsilon - p_\epsilon^h\|_{h,\Omega} \le C(\sqrt{\epsilon} + h + h_U), \tag{4.5}$$

where h and h_U are the mesh size of two families of partitions τ_h and τ_{h_U}.

Proof. For convenience, we define two useful solutions $u_\epsilon(u_h)$ and $p(u_\epsilon^h)$ by

$$\begin{cases} a(u_\epsilon(u_h), v) = (f + Bu_h, v), \quad \forall v \in X, \\ a(q, p_\epsilon(u_h)) = (g'(u_\epsilon(u_h)), q) \quad \forall q \in X. \end{cases} \tag{4.6}$$

and

$$a(q, p_\epsilon(u_\epsilon^h)) = (g'(u_\epsilon^h), q), \quad \forall q \in X. \tag{4.7}$$

Obviously, the solutions u_ϵ^h and p_ϵ^h are the standard finite element approximations of $u_\epsilon(u_h)$ and $p_\epsilon(u_\epsilon^h)$ of the elliptic homogenization problems. Thus, we deduce from (4.6)–(4.7) and (3.45) that

$$\begin{aligned}\|p_\epsilon(u_h) - p(u_\epsilon^h)\|_{h,\Omega} &\le C\|g'(u_\epsilon^h) - g'(u_\epsilon(u_h))\|_{0,\Omega} \\ &\le C\|u_\epsilon(u_h) - u_\epsilon^h\|_{0,\Omega} \\ &\le C\|u_\epsilon(u_h) - u_\epsilon^h\|_{h,\Omega} \le C(\sqrt{\epsilon} + h).\end{aligned} \tag{4.8}$$

From (4.2), (4.6), and $g'(y)$ being uniformly convex in H, it follows that

$$\|u_\epsilon - u_\epsilon(u_h)\|_{h,\Omega} \le C\|Bu - Bu_h\|_{0,\Omega} \le C\|u - u_h\|_{0,\Omega_U}, \tag{4.9}$$

$$\|p_\epsilon - p_\epsilon(u_h)\|_{h,\Omega} \le C\|g'(u_\epsilon) - g'(u_\epsilon(u_h))\|_{0,\Omega} \le C\|u - u_h\|_{0,\Omega_U}. \tag{4.10}$$

Thanks to (4.7), (4.3) and (3.44), we arrive at

$$\|p_\epsilon(u_\epsilon^h) - p_\epsilon^h\|_{h,\Omega} \le C(\sqrt{\epsilon} + h). \tag{4.11}$$

Combining (4.8), (4.10) with (4.11) gives

$$\begin{aligned}\|p_\epsilon - p_\epsilon^h\|_{h,\Omega} &\le \|p_\epsilon - p_\epsilon(u_h)\|_{h,\Omega} + \|p_\epsilon(u_h) - p_\epsilon(u_\epsilon^h)\|_{h,\Omega} + \|p_\epsilon(u_\epsilon^h) - p_\epsilon^h\|_{h,\Omega} \\ &\le C\|u - u_h\|_{0,\Omega_U} + C(\sqrt{\epsilon} + h).\end{aligned} \tag{4.12}$$

Similarly, we have

$$\begin{aligned}\|u_\epsilon - u_\epsilon^h\|_{h,\Omega} &\le \|u_\epsilon - u_\epsilon(u_h)\|_{h,\Omega} + \|u_\epsilon(u_h) - u_\epsilon^h\|_{h,\Omega} \\ &\le C\|u - u_h\|_{0,\Omega_U} + C(\sqrt{\epsilon} + h).\end{aligned} \tag{4.13}$$

As for the error estimate of the control function $\|u - u_h\|_{0,\Omega_U}$, we deduce from (4.4) that by utilizing the uniformly convex $h(u)$

$$C_0^2\|u - u_h\|_{0,\Omega_U}^2 \le (h'(u) - h'(u_h), u - u_h)_U. \tag{4.14}$$

Taking $w = w_h = v_h$ in (4.2) and (4.3), respectively, and using (4.14),we have

$$\begin{aligned}c\|u - u_h\|_{0,\Omega_U}^2 &\le (h'(u) - h'(u_h), u - u_h)_U \\ &\le (B^*p_\epsilon^h - B^*p_\epsilon, u - u_h)_U + (h'(u_h) + B^*p_\epsilon^h, v_h - u)_U \\ &\le (B^*p_\epsilon^h - B^*p_\epsilon(u_h), u - u_h)_U + (B^*p_\epsilon(u_h) - B^*p_\epsilon, u - u_h)_U \\ &\quad + (h'(u_h) - h'(u), v_h - u)_U + (h'(u) + B^*p_\epsilon, v_h - u)_U \\ &\quad + (B^*p_\epsilon(u_h) - B^*p_\epsilon, v_h - u)_U + (B^*p_\epsilon^h - B^*p_\epsilon(u_h), v_h - u)_U \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned} \tag{4.15}$$

Note that $\overline{v}$ is the average interpolation operator of $v \in H^{1,q}(\Omega)$ defined in [23]. For $i = 0, 1$ and $1 \le q \le \infty$, there holds

$$\|v - \overline{v}\|_{i,q,K} \le C \sum_{K \cap K'} h_K^{1-i}\|v\|_{1,q,K'}. \tag{4.16}$$

Thanks to the properties of the operator B^* and function $g(y)$, it is easy to find that

$$\begin{aligned}|I_1| &\le \|B^*p_\epsilon^h - B^*p_\epsilon(u_h)\|_{0,\Omega}\|u - u_h\|_{0,\Omega_U} \\ &\le C\|p_\epsilon^h - p_\epsilon(u_h)\|_{0,\Omega}\|u - u_h\|_{0,\Omega_U} \\ &\le C\|p_\epsilon^h - p_\epsilon(u_h)\|_{h,\Omega}^2 + \frac{C_0}{6}\|u - u_h\|_{0,\Omega_U}^2.\end{aligned} \tag{4.17}$$

By the definition of self-conjugated operator, uniformly convex, and the Young inequality, it follows that

$$\begin{aligned}|I_2| &= (B(u - u_h), p_\epsilon - p_\epsilon(u_h)) \\ &= a(u_\epsilon - u_\epsilon(u_h), p_\epsilon - p_\epsilon(u_h)) \\ &= (g'(u_\epsilon) - g'(u_\epsilon(u_h)), u_\epsilon - u_\epsilon(u_h)) \\ &\le -C\|u_\epsilon - u_\epsilon(u_h)\|_{0,U}^2 \le 0.\end{aligned} \tag{4.18}$$

Also, we have

$$\begin{aligned}|I_3| &\le \|h'(u_h)-h'(u)\|_{0,\Omega}\|u-\bar{u}\|_{0,\Omega_U}\\ &\le C\|u-\bar{u}\|^2_{0,\Omega_U}+\frac{C_0}{6}\|u-u_h\|^2_{0,\Omega_U}, \end{aligned}\tag{4.19}$$

$$\begin{aligned}|I_4| &\le ((h'(u)+B^*p_\epsilon)-\overline{(h'(u)+B^*p_\epsilon)},\bar{u}-u)_U\\ &\le Ch_U(\|u\|_{1,U}+\|p\|_{1,U})\|u-\bar{u}\|_{0,\Omega_U}\le ch_U^2, \end{aligned}\tag{4.20}$$

$$\begin{aligned}|I_5| &\le C\|p_\epsilon(u_h)-p_\epsilon\|_{0,\Omega}\|u-\bar{u}\|_{0,\Omega_U}\\ &\le C\|u-u_h\|_{0,\Omega_U}\|u-\bar{u}\|_{0,\Omega_U}\\ &\le \frac{C_0}{6}\|u-u_h\|^2_{0,\Omega_U}+C\|u-\bar{u}\|^2_{0,\Omega_U}, \end{aligned}\tag{4.21}$$

$$\begin{aligned}|I_6| &\le C\|p^h_\epsilon-p_\epsilon(u_h)\|_{0,\Omega}\|u-\bar{u}\|_{0,\Omega_U}\\ &\le C\|p^h_\epsilon-p_\epsilon(u_h)\|^2_{h,\Omega}+C\|u-\bar{u}\|^2_{0,\Omega_U}. \end{aligned}\tag{4.22}$$

Combining all these inequalities yields

$$\|u-u_h\|_{0,\Omega_U}\le C(\sqrt{\epsilon}+h_U+h).\tag{4.23}$$

Thus, using a triangle inequality, (4.12), (4.13) and (4.23), we obtain (4.5). □

Acknowledgements

The authors would like to thank the editor and referees for their valuable comments and suggestions which helped to improve this paper. The work was supported in part by the NSF of China (No. 10701001), and Natural Science Basic Research Plan in Shaanxi Province of China (Program No. SJ08A14).

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